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Z-TRANSFORM AND BERNOULLI POLYNOMIALS

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Abstract:

Ever since Swiss mathematician Jacob Bernoulli published his work on Bernoulli Numbers in 1713, there has been a great amount of work done regarding them. In this paper, we try to generalize the Bernoulli numbers to a class of polynomials called Bernoulli Polynomials. In particular, we see that the constant terms of the Bernoulli polynomials are precisely the Bernoulli numbers. In this paper, we employ the Z-transform technique to deduce a Cauchy convolution-type identity involving Bernoulli polynomials.

Key Words: Bernoulli Polynomials, Z-Transform, Cauchy Convolution-Type Identity, Generating Function.

1. Introduction:

Bernoulli numbers are numbers which occur as coefficients of $\frac{x^n}{n!}$ in the Taylor's series expansion of the function $\frac{x}{e^x-1}$ about x = 0

The nth Bernoulli polynomialis given by $B_n(t)=\sum_{k=0}^{n}\binom{n}{k}B_k t^{n-k}$ where B_k in the right hand side is the kth Bernoulli number.

If $B_n(y)$ and $B_n = B_n(0)$ are the numbers and polynomials of Bernoulli [1-3], respectively, then we construct the differences:

$$A_n(y) \coloneqq B_n(y) - B_n, \quad n \geq 0 \qquad \therefore \quad A_0(y) = 0, \quad A_1(y) = y, \dots, \tag{1}$$

And it is known the following Cauchy convolution-type identity [4-6]:

$$\sum_{k=0}^{n} x^k \frac{A_{k+1}\left(\frac{1}{x}\right)}{(k+1)!}\frac{A_{n-k+1}(x)}{(n-k+1)!} = 0, \qquad n \geq 1. \tag{2}$$

In Sec. 2 we use the Z-transform technique [7-10] to obtain (2).

2. Z-Transform Method:

From the generating function for Bernoulli polynomials [1] are immediate the Z-transforms:

$$Z\left\{\frac{B_n(y)}{n!}\right\} = \frac{e^{y/z}}{z\,(e^{1/z}-1)}, \qquad Z\left\{\frac{B_n}{n!}\right\} = \frac{1}{z\,(e^{1/z}-1)}, \tag{3}$$

Which can be applied to (1) to deduce the expressions:

$$Z\left\{\frac{A_{n+2}(x)}{(n+2)!}\right\} = z\left(\frac{e^{x/z}-1}{e^{1/z}-1} - x\right), Z\left\{x^n\frac{A_{n+1}\left(\frac{1}{x}\right)}{(n+1)!}\right\} = \frac{e^{1/z}-1}{e^{x/z}-1}, Z\left\{x^n\frac{A_{n+2}\left(\frac{1}{x}\right)}{(n+2)!}\right\} = \frac{z}{x}\left(\frac{e^{1/z}-1}{e^{x/z}-1} - \frac{1}{x}\right), \tag{4}$$

Therefore:

$$Z\left\{x^n\frac{A_{n+1}\left(\frac{1}{x}\right)}{(n+1)!}\right\} Z\left\{\frac{A_{n+2}(x)}{(n+2)!}\right\} = z\left(1 - x\,\frac{e^{1/z}-1}{e^{x/z}-1}\right) = -x^2\, Z\left\{x^n\frac{A_{n+2}\left(\frac{1}{x}\right)}{(n+2)!}\right\}, \tag{5}$$

Which is equivalent to the following convolution:

$$\sum_{k=0}^{m} x^{k+1}\frac{A_{k+1}\left(\frac{1}{x}\right)}{(k+1)!}\frac{A_{m-k+2}(x)}{(m-k+2)!} + x^{m+2}\frac{A_{m+2}\left(\frac{1}{x}\right)}{(m+2)!}\frac{A_1(x)}{1!} = 0, \quad m \geq 0, \tag{6}$$

And finally, with the change $n = m + 1$ the relation (6) implies (2), q. e. d.

Conclusion:

In this paper, we have applied the Z-transform to derive a Cauchy convolution-type identity involving Bernoulli polynomials. The proposed proof is concise and relies only on generating functions and elementary properties of the Z-transform. The method can potentially be extended to other classes of special polynomials, providing a unified framework for establishing similar convolution identities.

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